A quasi-isometric embedding theorem for groups

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Abstract

We show that every group H of at most exponential growth with respect to some left invariant metric admits a bi-Lipschitz embedding into a finitely generated group G such that G is amenable (respectively, solvable, satisfies a non-trivial identity, elementary amenable, of finite decomposition complexity, etc.) whenever H is. We also discuss some applications to compression functions of Lipschitz embeddings into uniformly convex Banach spaces, Følner functions, and elementary classes of amenable groups.

1 Introduction

It is well known that every countable group can be embedded in a group generated by 2 elements. This theorem was first proved by Higman, B.H. Neumann, and H. Neumann in 1949 [16] using HNN-extensions. Since then many alternative constructions have been found and the result has been strengthened in various ways. Most of the subsequent improvements were motivated by the desire to better control either the algebraic structure of the resulting finitely generated group or geometric properties of the embedding.

The original proof of the Higman-Neumann-Neumann theorem leads to "large" finitely generated groups even if one starts with a relatively "small" countable group; for instance, the resulting finitely generated group always contains non-abelian free subgroups. In the paper [21], B.H. Neumann and H. Neumann suggested an alternative approach based on wreath products, which allowed them to show that every countable solvable group can be embedded in a 2-generated solvable group. This approach was further developed by P. Hall [10] and used by Phillips [23] and Wilson [28] to prove analogous embedding theorems for torsion and residually finite groups.

In another direction, the Higman-Neumann-Neumann theorem was strengthened by the first author in [22]. Recall that a map $\ell\colon H \to \mathbb{N}\cup\{0\}$ is a *length function* on a group H if it satisfies the following conditions.

($\mathbf{L_1}$) $\ell(h) = 0$ iff $h = 1$.

($\mathbf{L_2}$) $\ell(h) = \ell(h^{-1})$ for any $g \in H$.

($\mathbf{L_3}$) $\ell(gh) \le \ell(g) + \ell(h)$ for any $g, h \in H$.

Further we say that the *growth of H with respect to ℓ is at most exponential* if there is a constant a such that for every $n \in \mathbb{N}$, we have

$$\#\{h \in H \mid \ell(h) \leq n\} \leq a^n \tag{1}$$

(this property was added to (L1)-(L3) in [22]). A length function $\ell \colon H \to \mathbb{N} \cup \{0\}$ defines a left-invariant metric on H by $\mathrm{d}(g,h) = \ell(g^{-1}h)$ and vice versa.

If G is a group generated by a finite set X and H is a (not necessary finitely generated) subgroup of G, the restriction of the wold length $|\cdot|_X$ to H obviously defines a length function on H and the growth of H with respect to $|\cdot|_X$ is at most exponential. In [22], the first author proved that any length function ℓ on a group H satisfying (1) can be realized up to bi-Lipschitz equivalence by such an embedding. The method of [22] is based on small cancellation techniques and consequently suffers from the same problem as the original Higman-Neumann-Neumann embedding: even if one starts with a "small" (say, abelian) group H, the resulting group G contains many free subgroups.

The goal of this paper is to suggest yet another construction which allows to control both the geometry of the embedding and the algebraic structure of the resulting finitely generated group. Given a group H, we denote by $\mathcal{E}(H)$ the class of all groups K such that every finitely generated subgroup of K embeds in a direct power of H. Our main result is the following.

Theorem 1.1. *Let H be a group of at most exponential growth with respect to a length function $\ell \colon H \to \mathbb{N} \cup \{0\}$. Then H embeds into a group G generated by a finite set X such that the following conditions hold.*

(a) There exists $c > 0$ such that for every $h \in H$, we have $c|h|_X \leq \ell(h) \leq |h|_X$.

(b) G has a normal series $G_1 \lhd G_2 \lhd G$, where G_1 is abelian and intersects H trivially, $G_2/G_1 \in \mathcal{E}(H)$, and G/G_2 is solvable of derived length at most 3.

In particular, our embedding allows to carry over a wide range of properties from the group H to the group G. For instance, we have the following.

Corollary 1.2. *In the notation of Theorem 1.1, if H is solvable (respectively, satisfies a non-trivial identity, elementary amenable, amenable, has property A, has finite decomposition complexity, uniformly embeds in a Hilbert space, etc.), then so is G.*

Recall that property A and groups of finite decomposition were introduced by Yu [29] and Guentner, Tessera, and Yu [15], respectively, with motivation coming from the Novikov conjecture and topological rigidity of manifolds. For definitions, properties, and applications we refer to [15, 24, 29] and references therein. Groups uniform embeddable in Hilbert spaces are discussed below. The corollary obviously follows from the theorem since the class of elementary amenable groups (respectively, amenable groups, countable groups with property A, countable groups of finite decomposition complexity, and countable groups

uniformly embeddable in Hilbert spaces) contain abelian groups and are closed with respect to the operations of taking direct unions, subgroups, and extensions [6, 7, 15, 20].

For amenable and elementary amenable groups, even the following fact was unknown. This is remarked by Gromov in [12, Section 9.3], where the reader can also find some potential applications of the existence of such an embedding.

Corollary 1.3. *Every countable elementary amenable (respectively, amenable) group can be embedded in a finitely generated elementary amenable (respectively, amenable) group.*

The proof of Theorem 1.1 is based on a modified version of the construction of P. Hall [10], which in turn goes back to B.H. Neumann and H. Neumann [21]. In general, this construction does not preserve elementary amenability, amenability, etc., as it uses unrestricted wreath products. A little improvement based on the existence of parallelogram-free subsets (see Definition 6) in finitely generated solvable groups fixes this problem. However the original construction does not allow to control the distortion of the embedding, so an essential modification is necessary to ensure (a). Our main technical tool here is a "metric" version of the Magnus embedding, which seems to be of independent interest (see Section 2).

One possible application of Theorem 1.1 is to constructing solvable, amenable, etc., groups with unusual geometric properties. In general, it is much easier to build an interesting geometry inside an infinitely generated group; then Theorem 1.1 allows to embed it in a finitely generated one. This philosophy has a few almost immediate implementations. We briefly discuss them below and refer to Section 3 for definitions and details.

Recall that to each map from a finitely generated group G to a metric space (S, d_S), one associates a non-decreasing *compression function* $\mathrm{comp}_f\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\mathrm{comp}_f(x) = \inf_{\mathrm{d}_X(u,v)\geq x} \mathrm{d}_S(f(u), f(v)),$$

where d_X is the word metric on G with respect to a finite generating set X.

Given two functions $r, s\colon \mathbb{R}_+ \to \mathbb{R}_+$, we write $r \preceq s$ if there exists $C > 0$ such that

$$r(x) \leq Cs(Cx) + C$$

for every $x \in \mathbb{R}_+$. As usual, $r \sim s$ if $r \preceq s$ and $s \preceq r$. Up to this equivalence, the compression function comp_f is independent of the choice of a particular finite generating set of G. If f is Lipschitz and satisfies $\mathrm{comp}_f(x) \to \infty$ as $x \to \infty$, then f is called a *uniform embedding.*

The study of group embeddings into Hilbert (or, more generally, Banach) spaces was initiated by Gromov in [11]. Motivated by his ideas, Yu [29] and later Kasparov and Yu [17] proved that finitely generated groups uniformly embeddable in a Hilbert space (respectively, a uniformly convex Banach space) satisfy the coarse Novikov conjecture. Another interesting result was proved by Guentner and Kaminker in [14]. They showed that if a finitely

generated group G admits a uniform embedding in a Hilbert space with compression $\succeq x^{\varepsilon}$ for some $\varepsilon > 1/2$, then the reduced C^*-algebra of G is exact; moreover, if the embedding is equivariant, then G is amenable. On the other hand, by a result of Brown and Guentner [5], any metric space of bounded geometry can be uniformly embedded into the ℓ^2- sum $\oplus l^{p_n}(\mathbb{N})$ for some sequence of numbers $p_n \in (1, +\infty)$, $p_n \to \infty$.

In [1], Arzhantseva, Drutu, and Sapir showed that for every function $\rho\colon \mathbb{R} \to \mathbb{R}$ satisfying $\lim\limits_{x\to\infty} \rho(x) = \infty$, there exists a finitely generated group G such that every Lipschitz map from G to a uniformly convex Banach space has compression function $\preceq \rho$. The groups constructed in [1] contain free subgroups and hence are not amenable. On the other hand, computations made in various papers (see, e.g., [2, 26] and references therein) suggest that for amenable groups the situation can be different.

For some time it was unknown even whether every finitely generated amenable group admits a Lipschitz embedding in a Hilbert space with compression function $\succeq x^{\varepsilon}$ for some $\varepsilon > 0$. This question was asked in [1, 2, 26] and answered negatively by Austin in [3]. Austin also remarks that his approach can probably be extended so far as to give a finitely generated amenable group G such that every Lipschitz map $f\colon G \to L_p$, $p \in [1, \infty)$, has compression function $\preceq \log x$. However his methods do not seem to alow to break this barrier and he asks whether every finitely generated amenable group G admits a Lipschitz embeddings $f\colon G \to L_p$ for every $p \in [1, \infty)$ with compression function $\succeq \log x$. We show that the answer is negative and, moreover, an analogue of the Arzhantseva-Drutu-Sapir result holds for amenable groups.

Corollary 1.4. *Let $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ be any function such that $\lim\limits_{x\to\infty} \rho(x) = \infty$. Then there exists a finitely generated elementary amenable group G such that for every Lipschitz embedding f of G into a uniformly convex Banach space, the compression function of f satisfies $\mathrm{comp}_f \preceq \rho$.*

The proof of the corollary is inspired by [1] and uses expanders constructed by Lafforgue [18].

Our approach can also be used to obtain some known results about Følner functions introduced by Vershik [27]. For a finitely generated amenable group G, its Følner function, $F\o l\colon \mathbb{N} \to \mathbb{N}$ measures the asymptotic growth of Følner sets of G. Vershik conjectured that there exist amenable groups G with the Følner functions growing arbitrary fast and his conjecture remained open for several decades until Erschler proved it in [9]. The groups constructed in [9] are of intermediate growth and, in particular, they are amenable but not elementary amenable [6].

Note that for many elementary amenable groups G, $F\o l_G$ is bounded from above by an iterated exponential function. For instance, this is true for all finitely generated solvable groups by the main result of [8]. On the other hand, the following was announced by Eschler in [8] and later proved by Gromov in [12]. Corollary 1.4 allows us to recover this result.

Corollary 1.5 (Erschler-Gromov)**.** *For any function $\sigma\colon \mathbb{N} \to \mathbb{N}$, there exists an elementary amenable group G with $F\o l_G \succeq \sigma$.*

The last application concerns the class EA of elementary amenable groups. Recall that every $G \in EA$ can be "constructed" from finite and abelian groups by taking subgroups, quotients, extensions, and direct unions. The *elementary class* of G, $\mathrm{c}(G)$, is an ordinal number that measures the complexity of this procedure (see Section 3 for the precise definition). It is easy to see that for every countable group $G \in EA$, one has $\mathrm{c}(G) < \omega_1$, where ω_1 is the first uncountable ordinal; however it is unclear how large $\mathrm{c}(G)$ can be. For finitely generated groups this question is mentioned by Gromov in [12, Section 9.3]. For subgroups of Thompson's group F it was also addressed by Brin in [4], where he showed that for every non-limit ordinal $\alpha \leq \omega^2 + 1$ there exists an elementary amenable subgroup $G \leq F$ such that $\mathrm{c}(G) = \alpha$. However Brin remarks that his approach does not allow to go beyond ω^2+1, at least for groups of orientation preserving piecewise linear self homeomorphisms of the unit interval.

In the corollary below we give a complete description of ordinals that can be realized as elementary classes of countable elementary amenable groups.

Corollary 1.6. *Let EA_c and EA_{fg} denote the sets of all countable elementary amenable and finitely generated elementary amenable groups, respectively. Then we have*

$$\mathrm{c}(EA_c) = \{\alpha + 1 \mid \alpha < \omega_1\} \cup \{0\}$$

and

$$\mathrm{c}(EA_{fg}) = \{\alpha + 2 \mid \alpha < \omega_1\} \cup \{0, 1\}.$$

We conclude with the following.

Problem 1.7. *Does every recursively presented countable amenable (or elementary amenable) group embed into a finitely presented amenable group?*

2 Proof of the main theorem

Preliminary information Let G be a group generated by a set $X \subseteq G$. Given an element $g \in G$, one defines the word length of g with respect to X, $|g|_X$, as the length of a shortest word in the alphabet $X \cup X^{-1}$ that represents g in G.

We recall the definition of the wreath product of two groups A and B. The *base subgroup* $\overline{W}$ is the set of functions from B to A with pointwise multiplication. The group B acts on $\overline{W}$ from the left by automorphisms, such that for $f \in W$ and $b \in B$, the function $b \circ f$ is given by

$$(b \circ f)(x) = f(xb) \quad for \quad every \quad x \in B$$

This action defines a semidirect product $\overline{W}B$ called the *Cartesian wreath product* of the groups A and B, denoted $A \,\mathrm{Wr}\, B$. Hence we have $bfb^{-1} = b \circ f$ in this group, and every element of $A \,\mathrm{Wr}\, B$ is uniquely factorized as fb, where $f \in \overline{W}, b \in B$

The functions $B \to A$ with finite support, i.e. with the condition $f(x) = 1$ for almost every $x \in B$, form a subgroup W in $\overline{W}$. Respectively, we have the subgroup WB of $A \operatorname{Wr} B$, denoted $A \operatorname{wr} B$ and called the *direct wreath product* of the groups A and B. Thus, the base W of $A \operatorname{wr} B$ is the direct product of the subgroups $A(b)$ $(b \in B)$ isomorphic to A, and $A(b) = bA(1)b^{-1}$ in $A \operatorname{wr} B$. One may identify the subgroup $A(1)$ with A, and so the wreath product is generated by the subgroups A and B.

If the group A is abelian, then so is W, and one may use the additive notation for the elements of the base subgroup W of $A \operatorname{wr} B$. The base W becomes a module over the group ring $\mathbb{Z}B$. In particular, if A is a free abelian group with basis $(a_1, \dots, a_n)$, then W is a free $\mathbb{Z}B$-module, and in the module notation, every element of W has a unique presentation as $t_1a_1 + \cdots + t_na_n$ with $t_1, \dots, t_n \in \mathbb{Z}B$.

Modified Magnus embedding. The standard Magnus homomorphism may be thought of as an embedding of a group of the form $F/[N,N]$ in the wreath product $A \operatorname{wr} F/N$, where $A \simeq F/[F,F]$. The goal of this section is to describe a modified version that also depends on a length function ℓ on F/N. Our main result in this direction is Lemma 2.1; its analogue for the standard version of the Magnus embedding, Corollary 2.2, also seems new.

Let H be a group with a set of generators $(h_i)_{i\in I}$ and ϵ be the homomorphism of a free group F with basis $(x_i)_{i\in I}$ onto H given by $x_i \mapsto h_i$. Denote by N the kernel of ϵ. Let A be a free abelian group with the basis $(a_i)_{i\in I}$ and let $V = A \operatorname{wr} H$. The *Magnus homomorphism* $\mu_\epsilon\colon F \to V$ maps x_i to a_ih_i $(i \in I)$. By Magnus's theorem [19], $\ker \mu_\epsilon = [N,N]$, the derived subgroup of N.

Every element of V has a unique form wg, where $g \in H$ and w belongs to the base subgroup W, and so $w = \sum_{i\in I} t_ia_i$, where $t_i \in \mathbb{Z}H$ and almost all t_i-s are 0. By the Remeslennikov – Sokolov criterion, if $wg \in \mu_\epsilon(F)$, then

$$\sum_{i\in I} t_i(h_i - 1) = g - 1$$

in $\mathbb{Z}H$. (We use only the easier half of 'iff' from [25], and for $wg = \mu(u)$, this equation is directly verified by the induction on the length of u.)

Assume that H is a group with a length function $\ell(h)$ and a set of generators $(h_i)_{i\in I}$. We set $l_i = \max(\ell(h_i), 1)$ define a modified Magnus homomorphism by the formula

$$\mu(x_i) = a_i^{l_i}h_i, \;\; i \in I.$$

Thus μ is a homomorphism of F to the subgroup isomorphic to the wreath product $A' \operatorname{wr} H$, where A' is the free abelian subgroup of A with basis $(l_ia_i)_{i\in I}$ (in additive notation). Therefore the Remeslennikov – Sokolov property looks now as follows. Let $w = \sum_{i\in I} t_ia_i$, where $t_i \in \mathbb{Z}H$, and let $g \in H$.

(RS) *If $wg \in \mu(F)$, then $t_i = l_is_i$ for some $s_i \in \mathbb{Z}H$ and $\sum_{i\in I} s_i(h_i - 1) = g - 1$ in $\mathbb{Z}H$.*

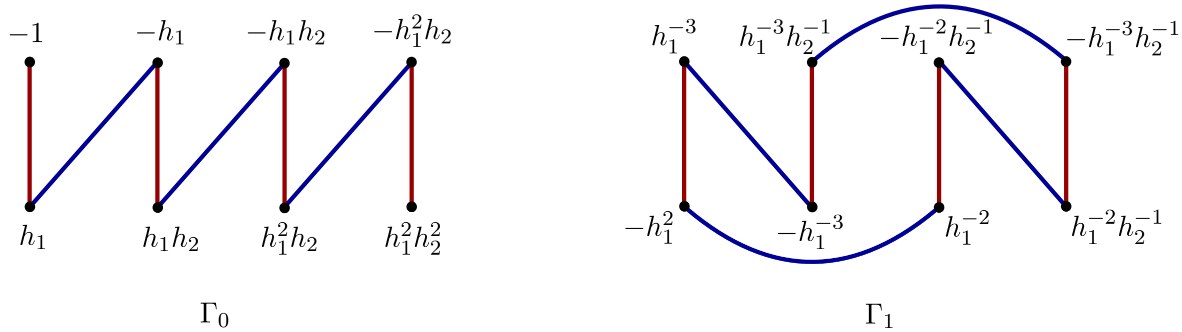

Figure 1: The cancellation graph for the image of the word $x_1^{-3}x_2^{-1}x_1x_2x_1^3x_2x_1x_2$ under the standard Magnus homomorphism.

For an element $t = \sum_{h\in H} k_h h$ of $\mathbb{Z}H$ ($k_h \in \mathbb{Z}$), we define its norm by

$$||t|| = \sum_{h\in H} |k_h|.$$

Further for $w = \sum_{i\in I} t_i a_i \in W$, we set

$$||w|| = \sum_{i\in I} ||t_i||.$$

Lemma 2.1. *If in the above notation $wg \in \mu(F)$, then $\ell(g) \leq ||w||$.*

Proof. We may assume that $g \neq 1$ since otherwise it is nothing to prove.

Every element s_i is a sum of the form $\sum_j \pm g_{ij}$, where $g_{ij} \in H$, and in this sum, an element of H can occur $|k_{ij}|$ times either with $+$ or with $-$ (but not with both signs). Therefore equality (RS) can be rewritten as

$$\sum_i (\sum_j \pm g_{ij})(h_i - 1) = g - 1 \tag{2}$$

Note that the right-hand side of (2) has only two terms. We define a labeled *cancellation graph* Γ reflecting the cancellations in the left-hand side. By definition, Γ has vertices $(i,j,1)$ labeled by $\mp g_{ij}$ and the vertices $(i,j,2)$ labeled by $\pm g_{ij}h_i$.

Every vertex $(i,j,1)$ is connected with $(i,j,2)$ by a red edge. So every vertex is connected by a red vertex with exactly one other vertex. Since almost all terms in the left-hand side of (2) cancel out, there is a pairing on the set of vertices without two vertices o and o' labeled by -1 and g, respectively, such that one of the vertex in each pair is labeled by some $x \in H$ and another one is labeled by $-x$. We fix the pairing and connect two vertices of a pair by a blue edge.

As an example, consider the free abelian group H with basis $\{h_1, h_2\}$. For simplicity take the standard word length ℓ on H with respect to the generators $\{h_1, h_2\}$. In this case our modified Magnus homomorphism $\mu\colon F \to A \operatorname{wr} H$, where $F = F(x_1, x_2)$ is the free group with basis $\{x_1, x_2\}$ and A is the free abelian group with basis $\{a_1, a_2\}$, coincides with the standard Magnus homomorphism defined by $\mu(x_i) = a_i h_i$ for $i = 1, 2$. Let

$$f = x_1^{-3} x_2^{-1} x_1 x_2 x_1^3 x_2 x_1 x_2$$

Then it is straightforward to compute

$$\mu(f) = \big((h_1h_2 + h_1^{-3}h_2^{-1} - h_1^{-3} + 1)a_1 + (h_1^2h_2 + h_1 + h_1^{-2}h_2^{-1} - h_1^{-3}h_2^{-1})a_2\big)h_1^2h_2^2.$$

Then (2) takes the form

$$\big(h_1h_2 + h_1^{-3}h_2^{-1} - h_1^{-3} + 1\big)(h_1 - 1) + \big(h_1^2h_2 + h_1 + h_1^{-2}h_2^{-1} - h_1^{-3}h_2^{-1}\big)(h_2 - 1) = h_1^2h_2^2 - 1.$$

The corresponding cancellation graph is drawn on Fig. 1.

Obviously every vertex of Γ belongs to exactly one red edge, and every vertex except for o, o', belongs to exactly one blue edge, i.e., o and o' have degree 1 while other vertices have degree 2. It follows that Γ decomposes into connected components $\Gamma_0, \Gamma_1, \ldots$, where Γ_0 is a simple arc connecting o and o' and other components are simple loops. The red edges and the blue ones must alternate in the directed path $o - o' = p = e_0 f_1 e_1 \ldots, f_d e_d$. Besides e_0 and e_d (and so all e_i-s) are red since no blue edges start/end in o or in o'.

Note that $||s_i||$ is equal to the number of vertices of the form $(i, j, 1)$ (with various j-s) in Γ, and $||w|| = \sum_i ||t_i|| = \sum_i l_i ||s_i||$. Therefore to obtain $||w||$, we may assign the weight l_i to every vertex $(i, j, 1)$ and sum these weights. Hence to estimate $||w||$ *from below*, it suffices to sum the assigned weights of the vertices only along the path p. We have $||w|| \geq \sum_k l_{i_k}$, over all vertices of the form $(i_k, j_k, 1)$ passed by p. Since every red edge e of p connects some $(i, j, 1)$ and $(i, j, 2)$, we can assign the weight l_i to such a red edge e. So

$$||w|| \geq \sum_{k=0}^{d} l_{i_k}, \tag{3}$$

where $l_{i_k} = \max(\ell(h_{i_k}), 1)$ is the weight of e_k.

Let $-x_{2k}$ and x_{2k+1} be the labels of the original and the terminal vertices of the red edge e_k in p. Using the definition of red and blue edges, we see for $k > 0$, that $x_{2k+1} = -x_{2k} h_{i_k}^{\pm 1} = x_{2k-1} h_{i_k}^{\pm 1}$. It follows that $\ell(x_{2k+1}) \leq \ell(x_{2k-1}) + \ell(h_{i_k})$. Likewise $\ell(x_1) \leq \ell(x_0) + \ell(h_{i_0})$. Hence by induction,

$$\ell(g) = \ell(x_{2d+1}) \leq \ell(x_0) + \sum_{k=0}^{d} \ell(h_{i_k}) \leq 0 + \sum_{k=0}^{d} l_{i_k}$$

since $\ell(x_0) = \ell(1) = 0$. This inequality and (3) prove the lemma. $\square$

Corollary 2.2. *Let H be a group with a (finite or infinite) set of generators $X = (h_i)_{i\in I}$ and μ is the Magnus homomorphism $x_i \mapsto a_i h_i$ $(i \in I)$ of the free group $F = F(X)$ to the wreath product $V = A\,wr\,H$ of a free abelian group A with a basis $(a_i)_{i\in I}$ and H. If $\mu(f) = wg$, where $f \in F$, $g \in H$, and w belongs to the base subgroup of V, then*

(a) $|g|_X \le ||w||$;

(b) $|g|_X = ||w||$ if and only if the canonical images of f in $H \simeq F/N$ and in $F/[N, N]$ have equal lengths with respect to X; here N is the kernel of the homomorphism $\epsilon\colon x_i \mapsto h_i$.

Proof. (a) The function $\ell(h) = |h|_X$ is a length function, and $\max(\ell(h_i), 1) = 1$ for every $h_i \in X$. So the modified Magnus homomorphism μ from Lemma 2.1 is just the standard Magnus homomorphism in this case. Hence the statement follows from Lemma 2.1.

(b) Let r be the length of f in $F/[N, N]$, that is, by Magnus' theorem, the length of $\mu(f)$ in $\mu(F)$ with respect to $(a_i h_i)_{i\in I}$. Thus $\mu(f) = y_1 \dots y_r$, where each y_j is of the form $(a_i h_i)^{\pm 1}$. The element g is the ϵ-image of f in H.

Assume first that $r = |g|_X$. We want to prove the equality $|g|_X = ||w||$, and by statement (a), it suffices to prove the inequality $||w|| \le r$. So it sufficient to show by induction that if $y_1 \dots y_{j-1} = w'g'$ with $g' \in H$, $w' \in V$, and $||w'|| \le j - 1$, then $y_1 \dots y_{j-1} y_j = w''g''$ with $||w''|| \le j$. But $w'g'(a_j h_j)^{\pm 1} = w'ka_j^{\pm 1}k^{-1}(g'(h_j)^{\pm 1})$, where $k = g'$ or $k = g'h_j^{-1}$. Therefore $||w''|| \le ||w'|| + ||ka_j^{\pm 1}k^{-1}|| \le j - 1 + 1 = j$, as required.

Now we assume that $|g|_X = ||w|| \ge 1$ and will use the notation and arguments of Lemma 2.1 (but with $l_i = 1$ and $s_i = t_i$). The number of red edges in the component Γ_0 is at most $r = |g|_X$ since $||w||$ is the number of red edges in the entire Γ. On the other hand, r is at least the number of red edges in Γ_0 since the difference of the lengths of g_{ij} and $g_{ij}h_i$ labeling (with signs) the ends of a red edge differ by at most 1, and such a difference is 0 for blue edges of the path p connecting the vertices labeled by 1 and g. Hence $r = d + 1$ and passing every red edge e_j directed in p from 1 to g we increase the length of the vertex label exactly by 1. (Besides, the graph Γ is connected.) Let the vertices of e_d be labeled by $\mp g_{ij}$ and by $\pm g_{ij}h_i$ for some i, j, where $\pm g_{ij}$ is a summand of s_i. So there are two cases: either (1) $g = g_{ij}$ and $|g_{ij}h_i|_X = r - 1$ or (2) $g = g_{ij}h_i$ and $|g_{ij}|_X = r - 1$.

Case (1) Let $g' = g_{ij}h_i$ and $w'g' = (wg)(a_i h_i)$, i.e., $w' = w(ga_i g^{-1})$. We can say using module notation that passing from w to w' we add the summand $g = g_{ij}$ to s_i. Since it annihilates with the summand $-g_{ij}$ of s_i, we have $||w'|| = ||w|| - 1 = r - 1$. Thus $||w'|| = |g'|_X = r - 1$, and by induction, the length of $w'g'$ in the generators $(a_k h_k)_{k\in I}$ is equal to $r - 1$. So the length of wg in these generators is $\le r - 1 + 1$, as required.

Case (2) Let $g' = g_{ij}$ and $w'g' = (wg)(a_i h_i)^{-1}$. Now $w' = w(g_{ij}a_i^{-1}g_{ij}^{-1})$, i.e., we add $-g_{ij}$ to annihilate it with the summand g_{ij} of s_i and complete the proof as in Case 1.

□

Consider again a group H with all nontrivial elements as generators and a length function

$\ell\colon H \to \mathbb{N} \cup \{0\}$. Let A be the free abelian group with basis $\{a_h \mid h \in H\backslash\{1\}\}$ and let

$$Y = \{a_h,\, a_h^{\ell(h)} h \mid h \in H\backslash\{1\}\} \subseteq A \,\mathrm{wr}\, H.$$

Obviously Y generates the wreath product

$$V = A \,\mathrm{wr}\, H. \tag{4}$$

Corollary 2.3. *For every non-trivial element $h \in H \le V$, we have $|h|_Y = \ell(h) + 1$.*

Proof. Note that $h = a_h^{-\ell(h)}(a_h^{\ell(h)} h)$, and therefore $|h|_Y \le 1 + \ell(h)$.

Assume now that $h = y_1 \dots y_r$, where $r = |h|_Y$ and $y_j \in Y^{\pm 1}$ for $j = 1, \dots, r$. Let us move all y_j's of the form $(a_h^{\ell(h)} h)^{\pm 1}$ to the right using conjugation. We obtain $h = uv$, where $v \in \mu(F)$ for the homomorphism $\mu\colon F \to V$ given by the rule $x_h \mapsto a_h^{\ell(h)} h$, and $u = y_1' \dots y_q'$ is the product of some conjugates of the factors y_j's having the form $a_h^{\pm 1}$. (We assume that the number of such factors in the factorization of h is q.) It follows that $||y_j'|| = 1$ since the norm does not change under conjugation.

As in Lemma 2.1, we have $v = wg$ in V, where $g = h$ (use the projection $V \to H$). We can apply the assertion of Lemma 2.1 since $\max(\ell(h), 1) = \ell(h)$ for $h \in H\backslash\{1\}$, and conclude that $\ell(h) \le ||w||$.

On the one hand, $h = uwg$, and so $uw = 1$ and $||v|| = ||w||$. On the other hand, $||v|| \le q$ since $||y_j'|| = 1$ for every y_j'. Hence $q \ge ||w||$. But since $h \ne 1$, we must also have at least one factor of the form $(a_h^{\ell(h)} h)^{\pm 1}$ in the product $y_1 \dots y_r$. Therefore,

$$|h|_Y = r \ge q + 1 \ge ||w|| + 1 \ge \ell(h) + 1$$

as required. $\square$

The main construction. Throughout this subsection we use the notation $[x, y] = xyx^{-1}y^{-1}$.

Consider the Cartesian wreath product $V \,\mathrm{Wr}\, \mathbb{Z}$, where V is defined by (4). Define a set U of functions $\mathbb{Z} \to V$ as the union of the following two sets U_1 and U_2: U_1 consists of all element $f_{1,h}$ of the base of $V \,\mathrm{Wr}\, \mathbb{Z}$ such that

$$f_{1,h}(n) = \begin{cases} 1, & \text{if n} \le 0 \\ a_h^{l(h)} h, & \text{if n} > 0 \end{cases}$$

and U_2 consists of all functions $f_{2,h}$ such that

$$f_{2,h}(n) = \begin{cases} 1, & \text{if n} \le 0 \\ a_h, & \text{if n} > 0 \end{cases}$$

Let t be a generator of $\mathbb{Z}$. For definiteness let $t = 1$. We denote by K the subgroup generated by $Z = U \cup \{t\}$ in the wreath product $V \operatorname{Wr} \mathbb{Z}$. Then $[t, f_{1,h}] = tf_{1,h}t^{-1}f_{1,h}^{-1}$, considered as a function $\mathbb{Z} \to V$, takes only one nontrivial value $a_h^{\ell(h)}h$ at 0. Similarly, $[t, f_{2,h}]$ takes only one nontrivial value a_h at 0. Since Y is the set of generators of V, the group V is isomorphic to the subgroup of K generated by these commutators (which can be identified with the elements of Y) and canonically embedded into $V \operatorname{Wr} \mathbb{Z}$.

Lemma 2.4. *For any $h \in H$, we have $\ell(h) \le |h|_Z$.*

Proof. Let L denote the base of the wreath product $V \operatorname{Wr} \mathbb{Z}$ and let $\pi\colon L \to V$ be the projection which to every function $f\colon \mathbb{Z} \to V$ assigns its value $f(0)$. Fix $h \in H$. Let w be a word in the alphabet $Z^{\pm 1}$ representing h and such that $|h|_Z = |w|$. Applying the standard rewriting process to w we obtain the equality

$$h = (t^{\alpha_1} f_1^{\pm 1} t^{-\alpha_1}) \cdots (t^{\alpha_m} f_m^{\pm 1} t^{-\alpha_m}), \tag{5}$$

where $f_1, \ldots, f_m \in U$ and $m \le |h|_Z$. Now applying π to both sides of (5) we obtain $h = y_1^{\pm 1} \cdots y_m^{\pm 1}$, where $y_i \in Y \cup \{1\}$ because every value of a function from U (and of a conjugate of it by a power of t) is either trivial or belongs to Y. This and Corollary 2.3 imply the inequalities $l(h) \le |h|_Y \le m \le |h|_Z$. $\square$

The following concept is important for describing the algebraic structure of the finitely generated group from Theorem 1.1.

Definition 2.5. Let u_1, u_2, u_3, u_4 be elements of a group G such that $u_i \ne u_{i+1}$, for arbitrary i taken modulo 4. We say that the configuration (u_1, u_2, u_3, u_4) is a *parallelogram* if

$$u_1 u_2^{-1} u_3 u_4^{-1} = 1 \tag{6}$$

in G. A subset P of a group G is *parallelogram-free* if it contains no parallelograms.

Lemma 2.6. *Let G be a group, P a subset of G.*

(1) For every $g \in G$, if P is parallelogram-free so are the sets gP and Pg.

(2) The following properties are equivalent:

(a) P is parallelogram-free.

(b) $\#\{P \cap gP\} \le 1$ for every nontrivial $g \in G$.

(c) $\#\{P \cap Pg\} \le 1$ for every nontrivial $g \in G$.

Proof. Claim (1) is true since the condition (6) is invariant under simultaneous multiplication of all the elements $u_1, \ldots, u_4$ by g from the left or from the right.

To prove (2) we note that if (u_1, u_2, u_3, u_4) is a parallelogram in P, then two distinct elements u_1 and u_2 belong to both P and gP for non-trivial $g = u_1 u_4^{-1}$ since $gu_4 = u_1$

and $gu_3 = u_1u_4^{-1}u_3 = u_2$ by (6). Therefore (b) implies (a). Conversely, (a) implies (b) since given two distinct elements u_1 and u_2 in $\#\{P \cap gP\} \leq 1$, we have the parallelogram $(u_1, g^{-1}u_1, g^{-1}u_2, u_2)$ in P. Likewise we obtain the equivalence of (a) and (c) since the equation (6) is equivalent to each of the other 7 equations obtained from the left-hand side by inversion and cyclic permutations. □

Definition 2.7. Recall that a subset P of a group M with a finite set of generators S has exponential growth if there exist constants $c > 1$ and $\lambda > 0$ such that for every $n \in \mathbb{N}$,

$$\#\{w \in P \mid \lambda|w|_S < n+1\} \geq c^n$$

Lemma 2.8. *There exists a finitely generated metabelian group M with a parallelogram-free subset P of exponential growth.*

Proof. Take $M = \mathbb{Z} \operatorname{wr} \mathbb{Z}$ and $S = \{x_0, y_0\}$, where x_0 and y_0 generate the wreathed infinite cyclic groups. It is well known and easy to prove that the group M has exponential growth: every ball B_r of radius $r \geq 0$ centered at 1 has at least k^r elements for some $k > 1$. We will construct P as $P = \bigcup_{r=0}^{\infty} P_r$, where $P_0 = \{1\}$ and for $r > 0$, P_r is a maximal parallelogram-free extension of P_{r-1} in B_r. To prove the lemma, it suffices to show that $\#P_r > \frac{1}{2}k^{r/3}$ for every $r \geq 0$. In turn, it will be sufficient to prove that if N is a parallelogram-free subset of B_r and $n = \#N \leq \frac{1}{2}k^{r/3}$, then for some $x \in B_r \backslash N$, the subset $N' = N \cup \{x\}$ is also parallelogram-free.

We will require that x satisfies none of the equations (7–9) below.

$$x = u, \quad \text{where} \quad u \in N. \tag{7}$$

The total number of solutions of all such equations is n.

$$xu^{-1}vw^{-1} = 1 \quad \text{where} \quad u, v, w \in N. \tag{8}$$

The total number of solutions is at most n^3.

$$xu^{-1}xv^{-1} = 1, \quad \text{where} \quad u, v \in N. \tag{9}$$

Equation (9) is equivalent to the equation $(xu^{-1})^2 = vu^{-1}$. To prove that the total number of solutions of all such equations is at most n^2, it suffices to prove that any equation of the form $y^2 = a$ has at most one solution in M. Indeed, let $(w_1g)^2 = (w_2h)^2$ where g, h belong to the active infinite cyclic group and w_1, w_2 belong to the base subgroup of the wreath product M. Then we immediately have $g^2 = h^2$, and so $g = h$. Now we obtain $(1+g)w_1 = (1+g)w_2$ in the module notation. But the free module over the group ring of an infinite cyclic group has no module torsion, whence $w_1 = w_2$.

Now since

$$k^r > \frac{1}{2}k^{r/3} + \frac{1}{4}k^{2r/3} + \frac{1}{8}k^r \geq n + n^2 + n^3,$$

there is $x \in B_r$ such that x does not satisfy any of the equations (7–9). Note that the set $N' = N \cup \{x\}$ is bigger then N since x satisfies no equation of the form (7). The set N' is parallelogram-free because N is such, and x is not a solution of any of the equations (8-9). This completes the proof of the lemma. $\square$

Let now M be a metabelian group with a subset P provided by Lemma 2.8, S a finite generating set of M. By Lemma 2.6 (1), we may shift P and assume that $1 \in P$. Decreasing the constant λ in Definition 2.7 if necessary, we can assume that $c \geq \max\{a, 3\}$, where a and c are constants from (1) and Definition 2.7, respectively. Thus there exists a subset $P_0 \subseteq P$ such that $1 \in P_0$ and for every $n \geq 0$,

$$\#\{w \in P_0 \mid \lambda |w|_S < n+1\} = 2\#\{h \in H \backslash \{1\} \mid \ell(h) \leq n\} + 1.$$

We list all elements of $P_0 = \{1, w_1, w_2, \ldots\}$ and $U \cup \{1\} = \{1, u_1, u_2, \ldots\}$ in such a way that

$$\lambda |w_i|_S < \ell(h) + 1 \tag{10}$$

if $u_i = a_h^{\ell(h)} h$ or $u_i = a_h$. Let B denote the base of the wreath product $K \operatorname{Wr} M$. Let $g \colon M \to K$ be the element of B such that

$$g(x) = \begin{cases} t, & \text{if } x = 1, \\ u_i, & \text{if } x = w_i \text{ for some } i \in \mathbb{N}, \\ 1, & \text{if } x \notin P_0. \end{cases}$$

Let G be the subgroup of $K \operatorname{Wr} M$ generated by the finite set $X = S \cup \{g\}$.

Lemma 2.9. *(a) For any $r \in M$, the support of the function $[g, rgr^{-1}] \colon M \to K$ consists of at most one element and its value at this element is a commutator in K.*

(b) The group G contains all functions $M \to V$ mapping nontrivial elements to 1.

Proof. (a) We may assume that $r \neq 1$. Let $u \in M$. Then

$$[g, rgr^{-1}](u) = [g(u), (rgr^{-1})(u)] = [g(u), g(ur)]$$

Assume that this commutator is not trivial. Then both $g(u)$ and $g(ur)$ are nontrivial, and so both u and ur belong to $P_0 \subset P$ by the definition of g. If $[g, rgr^{-1}](u') = 1$ for $u' \neq u$, then u' and $u'r$ are also elements of P, and so $\#(P \cap Pr) \geq 2$ contrary Lemma 2.6 (2). Thus part (a) is proved.

(b) Recall that every generator $y \in Y \backslash \{1\}$ of V is identified in with a commutator $[t, u_i]$ in K for some $u_i \in U$. Therefore

$$[g, w_i g w_i^{-1}](1) = [g(1), (w_i g w_i^{-1})(1)] = [g(1), g(w_i)] = [t, u_i] = y \tag{11}$$

while $[g, w_i g w_i^{-1}](x) = 1$ whenever $x \neq 1$ by part (a). $\square$

By Lemma 2.9, the group V (and therefore H) can be regarded as a subgroup of G.

Lemma 2.10. *There is a positive constant θ such that $\theta|h|_X \leq \ell(h) \leq |h|_X$ for every element h of H.*

Proof. We may assume that $h \neq 1$. Recall that $h = a_h^{-l(h)}(a_h^{l(h)}h) = (y')^{-\ell(h)}y$ for some $y, y' \in Y$. Thus h is a product of $1 + l(h)$ generators from $Y^{\pm 1}$. In turn (see Lemma 2.9 (a) and (11)), $y = [g, w_i g w_i^{-1}]$ for some i, and hence y has length $\leq 4 + 4|w_i|_S$ with respect to $X = S \cup \{g\}$. Similarly, $y' = [g, w_{i'} g w_{i'}^{-1}]$ for some i', but the same computation as in Lemma 2.9, shows that

$$(y')^{-\ell(h)} = [g, w_{i'} g^{-\ell(h)} w_{i'}^{-1}]$$

in G, and so the length of $(y')^{-l(h)}$ with respect to X does not exceed $2\ell(h)+4|w_{i'}|_S+2$. Since by the choice of w_i and $w_{i'}$, their S-lengths (and X-lengths) do not exceed $\lambda^{-1}(\ell(h)+1)$ (see (10), we have

$$|h|_X \leq (4 + 4\lambda^{-1}(\ell(h)+1)) + (2\ell(h) + 4\lambda^{-1}(\ell(h)+1) + 2) \leq (8 + 16\lambda^{-1})\ell(h).$$

So the first inequality is proved with $\theta = (8 + 16\lambda^{-1})^{-1}$.

To prove the second inequality, we note that every $h \in H$, considered as an element of G, can be written as

$$h = p_0 g^{\alpha_1} p_1 \cdots g^{\alpha_k} p_k$$

for some $p_0, \ldots, p_k \in M$ and some integers $\alpha_1, \ldots, \alpha_k$ such that

$$|h|_X = \sum_{j=0}^{k} |p_k|_S + \sum_{i=1}^{k} |\alpha_j|. \tag{12}$$

Since h belongs to the base B, we have $p_0 \cdots p_k = 1$. Thus we can rewrite h in the form

$$h = r_1 g^{\alpha_1} r_1^{-1} \cdots r_k g^{\alpha_k} r_k^{-1}, \tag{13}$$

where $r_1 = p_0$, $r_2 = p_0 p_1$, $\ldots$, $r_k = p_0 p_1 \cdots p_{k-1} = p_k^{-1}$. Applying, to both sides of (13), the projection $B \to K$ that maps each $b \colon M \to K$ to $b(1)$, we obtain the equality

$$h = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

in the group K, where $x_j = (r_j g r_j^{-1})(1) = g(r_j)$. Note that x_j is equal to either 1 or $t \in Z$, or some $u \in U \subset Z$. Therefore $|h|_Z \leq \sum_j |\alpha_j| \leq |h|_X$ by (12). Finally, by Lemma 2.4, $l(h) \leq |h|_Z \leq |h|_X$. $\square$

Given a group H, we denote by $\mathbf{D}H$ the class of finite direct powers of H. Further let $\mathbf{SD}H$ be the class of all subgroups of groups from $\mathbf{D}H$. Finally let $\mathcal{E}(H) = \mathbf{LSD}H$ be the class of all groups which are locally in $\mathbf{SD}H$, i.e., $G \in \mathcal{E}(H)$ if every finitely generated subgroup of G belongs to $\mathbf{SD}H$.

Lemma 2.11. *Let T be a set, H a group, and $(f_i)_{i\in I}$ an arbitrary set of functions $T \to H$, where every f_i has a finite range in the group H. Then the subgroup $\langle f_i \mid i \in I\rangle$ of the Cartesian power H^T belongs to the class $\mathcal{E}(H)$.*

Proof. It suffices to proof that for every finite subset $J \subset I$, $\langle f_i \mid i \in J\rangle \in \mathbf{SD}H$.

Since every f_i has a finite range and J is also finite, there is a finite partition $T = \bigcup_{k=1}^r T_k$ such that the restriction of f_i to T_k is a constant for every $i \in J$ and every $k \le r$. It follows that every function f of the subgroup $G_J = \langle f_i \mid i \in J\rangle$ is a constant on every particular T_k. Therefore the mapping $f \mapsto (f(T_1), \dots, f(T_r))$ is an injective homomorphism from G_J to a direct product of r copies of H. □

Proof of Theorem 1.1. We have constructed the embeddings $H \hookrightarrow V \hookrightarrow G$. By Lemma 2.10, the embedding of H in G satisfies condition (a) of Theorem 1.1 with $c_1 = \theta$ and $c_2 = 1$. It remains to prove (b).

Note that if A is a normal abelian subgroup of a group C, then all functions with values in A form a normal abelian subgroup in arbitrary wreath product $C \operatorname{Wr} D$. Using this observation and the constructions of V, K, and G as subgroups of the corresponding wreath products, we see that the free abelian subgroup $A = \langle a_h \mid h \in H\backslash\{1\}\rangle$ is contained in a normal abelian subgroup G_1 of G such that the canonical image $\bar{V}$ of V in $\bar{G} = G/G_1$ is equal to $\bar{H}$ (the image of H in $\bar{G}$) and so it is isomorphic to H. Further to obtain the image $\bar{G}$ one should replace all a_h's by the identity in the definitions of the functions from the set U and, respectively, in the definition of the function g. (So the image $\bar{g}$ of g in $\bar{G}$ takes values in $\bar{K}$ which is isomorphic to a subgroup of $\bar{V} \operatorname{Wr} \mathbb{Z} \simeq H \operatorname{Wr} \mathbb{Z}$.)

Let R/G_1 be the normal closure of $\bar{g}$ in $\bar{G}$. Obviously $G/R \simeq M$, the metabelian group. So for $G_2 = [R, R]G_1$, we have that the quotient group G/G_2 is solvable of the derived length ≤ 3.

The group $\bar{R} = R/G_1$ is generated by all conjugates $r\bar{g}r^{-1}$, where $r \in M$. Therefore the group $\bar{G}_2 = G_2/G_1$ is generated by all the elements $d[\bar{g}, r\bar{g}r^{-1}]d^{-1}$, where $r \in M$ and $d \in \bar{G}$. It follows from Lemma 2.9 (a) that $[\bar{g}, r\bar{g}r^{-1}]$, and so $d[\bar{g}, r^{-1}\bar{g}r]d^{-1}$, considered as a function $M \to \bar{K}$, is either trivial or takes a nontrivial value at exactly one element of M and this value is an element of $[\bar{K}, \bar{K}]$. Now by Lemma 2.11, we have $\bar{G}_2 \in \mathcal{E}([\bar{K}, \bar{K}])$.

The group $\bar{K}$ is generated by $\bar{t}$ and the functions $\bar{f}_h = \bar{f}_{1,h}$ taking values in $\bar{H} \simeq H$. Therefore the group $[\bar{K}, \bar{K}]$ is generated by all $d[\bar{t}, \bar{f}_h]d^{-1}$ and $d[\bar{f}_h, \bar{f}_{h'}]d^{-1}$, where $d \in \bar{K}$. Each of the functions f_h has finite range in H. So do the commutators $[\bar{t}, \bar{f}_h]$ and $[\bar{f}_h, \bar{f}_{h'}]$. The same property holds for their conjugates by any element d since d is a finite product, where every factor is either $\bar{t}^{\pm 1}$ or $\bar{f}_g^{\pm 1}$ (with finite range) for some $g \in H$. Therefore $[\bar{K}, \bar{K}] \in \mathcal{E}(H)$ by Lemma 2.11. Since $G_2/G_1 = \bar{G}_2 \in \mathcal{E}([\bar{K}, \bar{K}])$, we have $G_2/G_1 = \mathcal{E}(H)$, and part (b) of the theorem is proved. □

3 Applications

Compression functions of Lipschitz embeddings in uniformly convex Banach spaces. Recall that the *compression function* $\mathrm{comp}(f)\colon \mathbb{R}_+ \to \mathbb{R}_+$ of a map f from a metric space (X, d_X) to a metric space (Y, d_Y) f is defined by

$$\mathrm{comp}_f(x) = \inf_{\mathrm{d}_X(u,v)\geq x} \mathrm{d}_Y(f(u), f(v)).$$

We start by summarizing some essential features of Lafforgue's construction of expanders [18].

Lemma 3.1. *There exists an infinite group Γ with a finite generating set X and a sequence of finite index normal subgroups*

$$\Gamma \rhd N_1 \rhd N_2 \rhd \dots$$

with trivial intersection such that the following holds. Let (G_k, d_k) denote the quotient group Γ/N_k endowed with the word metric corresponding to the image of the generating set X. Let E be a uniformly convex Banach space with a norm $\|\cdot\|$. Then there exist constants $R, \kappa > 0$ such that for every 1-Lipschitz embedding $f\colon (G_k, \mathrm{d}_k) \to (E, \|\cdot\|)$ one can find two elements $u, v \in G_k$ satisfying

$$\|f(u) - f(v)\| \leq R$$

and

$$\mathrm{d}_k(u, v) \geq \kappa \mathrm{diam}(G_k),$$

where $\mathrm{diam}(G_k)$ is the diameter of G_k with respect to d_k.

In fact, one can take Γ to be any co-compact lattice in $SL_3(F)$ for a non-archimedean local field F [18]. The proof of the lemma can be found in Section 3 of [1] (see Corollary 3.5 there) and is quite elementary modulo Lafforgue's paper.

We will also need the following property.

Lemma 3.2. *Let G be group generated by a finite set X. Let $G_1, G_2, \dots$ be a sequence of quotients of G. Denote by d_k the word metric on G_k with respect to the image of the set X. Let $H = \prod_{k=1}^{\infty} G_k$ be the direct product of G_k's. For an element $h = (g_k) \in \prod_{k=1}^{\infty} G_k$ we define*

$$\ell(h) = \sum_{k=1}^{\infty} k \mathrm{d}_k(1, g_k). \tag{14}$$

Then ℓ is a length function on H and the growth of H with respect to ℓ is at most exponential.

Proof. The fact that ℓ is a length function is obvious. To show that the growth of H with respect to ℓ is at most exponential it suffices to deal with the case when $G_1 \cong G_2 \cong \dots \cong G$. In the latter case, the map $H \to G \,\mathrm{wr}\, \mathbb{Z}$ that sends G_k to $t^k G t^{-k}$, where t is a generator

of $\mathbb{Z}$, extends to a Lipschitz embedding of H in $G \operatorname{wr} \mathbb{Z}$ because the length of the image of $h = (g_k)$ in $G \operatorname{wr} \mathbb{Z}$ does not exceed $\sum(2k + d(1, g_k))$ over all $g_k \neq 1$, which does not exceed $\sum_{k=1}^{\infty} k\mathrm{d}_k(1, g_k) = 3\ell(h)$. Since the wreath product is finitely generated, the claim is proved. $\square$

Proof of Corollary 1.4. Fix any function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to\infty} \rho(x) = \infty$. It suffices to prove the corollary for the function $\rho'(x) = \inf_{t\in[x,\infty)} \rho(t)$. Hence we can assume that ρ is non-decreasing without loss of generality. Further let $\mathcal{G} = \{(G_k, \mathrm{d}_k)\}$ be the family of finite groups provided by Lemma 3.1 and let $D_k = \operatorname{diam}(G_k)$ denote the diameter of G_k with respect to d_k. Since $D_k \to \infty$ as $k \to \infty$, passing to a subsequence of $\mathcal{G}$ if necessary we can assume that

$$\rho(D_k) \geq k \tag{15}$$

for all $k \in \mathbb{N}$.

Let $H = \prod_{k=1}^{\infty} G_k$ be the direct product of all G_k and let $\ell\colon H \to \mathbb{N}\cup\{0\}$ be the function on H defined by (14). Then, by Lemma 3.2, ℓ is a length function on H and the growth of H with respect to ℓ is at most exponential. Note that H is elementary amenable. Hence by Theorem 1.1, there exists an elementary amenable group G containing H and generated by a finite set X and a constant $c > 0$ such that for every $h \in H$, we have

$$c|h|_X \leq \ell(h) \leq |h|_X. \tag{16}$$

Let f be a Lipschitz embedding of G in a uniformly convex Banach space E, L its Lipschitz constant. Note that $\mathrm{comp}_{\frac{1}{L}f} \sim \mathrm{comp}_f$. Thus we can assume that f is 1-Lipschitz. By (16) and (14), for every $u, v \in G_k \leq G$ we have

$$\|f(u) - f(v)\| \leq \mathrm{d}_X(u, v) \leq \ell(u^{-1}v)/c = k\mathrm{d}_k(u, v)/c.$$

Thus the embedding $G_k \to G$ composed with f gives us a (k/c)-Lipschitz map $(G_k, d_k) \to E$. Rescaling and applying Lemma 3.1, we can find two elements $u, v \in G_k$ such that

$$\|f(u) - f(v)\| \leq kR/c \tag{17}$$

and

$$\mathrm{d}_X(u, v) \geq \ell(u^{-1}v) \geq \mathrm{d}_k(u, v) \geq \kappa D_k. \tag{18}$$

By the definition of the compression function, the inequalities (17) and (18) imply

$$\mathrm{comp}_f(\kappa D_k) \leq kR/c.$$

Finally, for any large enough $x \in \mathbb{R}_+$, we have $\kappa D_{k-1} \leq x \leq \kappa D_k$ for some $k \geq 2$. Using (15) and the fact that ρ is non-decreasing, we obtain

$$\mathrm{comp}_f(x) \leq \mathrm{comp}_f(\kappa D_k) \leq \frac{kR}{c} \leq \frac{2(k-1)R}{c} \leq \frac{2R\rho(D_{k-1})}{c} \leq \frac{2R}{c}\rho\left(\frac{x}{\kappa}\right).$$

Thus $\mathrm{comp}_f \preceq \rho$. $\square$

Følner functions. We recall the definition of a Følner function of an amenable group introduced by Vershik [27]. A finite subset A of a finitely generated group G is ε- Følner (with respect to a fixed finite generating set X of G) if

$$\sum_{x \in X} |Ax \triangle A| \leq \varepsilon |A|,$$

where $\triangle$ denotes symmetric difference. The Følner function of an amenable group G (with respect to X) is defined by

$$F\o l_{X,G}(n) = \min\{|A| : A \subseteq G \text{ is a } 1/n - \text{Følner w.r.t. } G\}.$$

Up to the standard equivalence relation induced by $\preceq$, $F\o l_{X,G}$ is independent of the choice of a finite generating set X. The corresponding equivalence class is denoted by $F\o l_G$.

Let $\mathcal{C}$ be a class of groups. By a *rank function* ρ on $\mathcal{C}$ we mean a map $\rho \colon \mathcal{C} \to P$, where P is a poset, such that the following conditions hold.

($\mathbf{R_1}$) For every $G \in \mathcal{C}$ and every $H \leq G$, if $H \in \mathcal{C}$ then $\rho(H) \leq \rho(G)$.

($\mathbf{R_2}$) For every $G \in \mathcal{C}$ and every quotient group Q of G, if $Q \in \mathcal{C}$ then $\rho(Q) \leq \rho(G)$.

A rank function $\rho \colon \mathcal{C} \to P$ is *unbounded* if $\rho(\mathcal{C})$ has no largest element elements.

Example 3.3. (a) The derived length is an unbounded rank function on the class of solvable groups.

(b) Similarly the function c defined below is a rank function on the class of elementary amenable groups which takes values in ordinal numbers. Indeed the conditions ($\mathbf{R_1}$) and ($\mathbf{R_2}$) follow from Lemma 3.6. If we restrict our attention to the set of countable elementary amenable groups, then Corollary 1.6 shows that c is unbounded.

Recall that a group G is *SQ-universal* if every countable group can be embedded into a quotient of G. The Higman-Neumann-Neumann theorem discussed in the introduction may be restated as follows: the free group of rank 2 is SQ-universal. Given a class of groups $\mathcal{C}$, we say that a group $G \in \mathcal{C}$ is *SQ-universal group in the class* $\mathcal{C}$ if every countable group from C embeds in a quotient of G.

Lemma 3.4. *Let $\mathcal{C}$ be a class of countable groups with an unbounded rank function.*

(a) There is no SQ-universal group in the class $\mathcal{C}$.

(b) If, in addition, every countable family of groups from $\mathcal{C}$ embeds simultaneously into a group from $\mathcal{C}$, then for any $\sigma \in \rho(\mathcal{C})$, there exists an uncountable chain in $\rho(\mathcal{C})$ with minimal element σ.

Proof. The first statement is obvious from the definition. To prove (b) we first note the following.

($*$) Every countable subset of $\rho(C)$ has an upper bound.

Indeed given a countable subset $c \in \rho(C)$, consider the groups $G_1, G_2, \ldots$ from $\mathcal{C}$ such that $\rho(\{G_1, G_2, \ldots\}) = c$. Let $G \in \mathcal{C}$ be a group which contains all G_i's. Then $\rho(G)$ is an upper bound for c.

Suppose now that every chain in $\rho(\mathcal{C})$ with minimal element σ is countable. Then ($*$) and the Zorn Lemma imply that the set $\{\tau \in \rho(\mathcal{C}) \mid \tau \succeq \sigma\}$ contains a maximal element μ. Using ($*$) again, we conclude that μ the largest element in $\rho(\mathcal{C})$, which contradicts our assumption. □

Let $F\o l$ denote the function which maps a finitely generated amenable group G to the $F\o l_G$. Note that the relation $\preceq$ defined in the introduction is a preorder, and hence induced a partial order the set of corresponding equivalence classes. We denote this order also by $\preceq$. It was proved by Erschler in [8] that with respect to $\preceq$, $F\o l$ is a rank function on the class of finitely generated amenable groups. In another paper [9], she also showed that $F\o l$ is unbounded on the class of finitely generated amenable groups. Moreover, is is unbounded on the class of finitely generated groups of subexponential growth. As a consequence of Corollary 1.4, we obtain below that $F\o l$ is unbounded on the class of elementary amenable groups.

Proof of Corollary 1.5. There is a standard construction of a uniform embedding of amenable groups into a Hilbert space, where the compression of the embedding is controlled by the Følner functions (see, e.g., [29]). That is, if $F\o l$ was bounded on the class of finitely generated elementary amenable groups, there would exist Lipschitz embedding of every finitely generated elementary amenable group into a Hilbert space with compression function $\succeq \rho$ for some ρ satisfying conditions of Corollary 1.4. A contradiction. □

Remark 3.5. Lemma 3.4 also implies that for every $\sigma \colon \mathbb{N} \to \mathbb{N}$, there is an uncountable chain of Følner functions of elementary amenable groups bounded by σ from below. As a corollary of Lemma 3.4 and Corollary 1.5, we also obtain that, unlike in the class of all groups, there are no SQ-universal groups in the classes of elementary amenable groups and amenable groups.

Elementary classes. Recall that the class of amenable groups is closed under the following operations [20]:

(S) Taking subgroups.

(Q) Taking quotients.

(E) Taking extensions.

(U) Taking direct unions.

The class of elementary amenable groups EA was defined by Chou [6] as the smallest class containing all finite and abelian groups and closed under the four operations (S)–(U). Alternatively, one can define EG inductively as follows. Let EG_0 be the class of all finite and abelian groups. Let α be an ordinal. If α is limit, define $EG_\alpha = \bigcup_{\beta<\alpha} EG_\beta$. If α is a successor ordinal, let EG_α be the class of groups that can be obtained from groups in $EG_{\alpha-1}$ by applying (E) or (U) once. The following lemma summarizes some results of [6].

Lemma 3.6 (Chou). *(a) For every ordinal α, EG_α is closed under (S) and (Q).*

(b) $EA = \bigcup_\alpha EG_\alpha$, where the union is taken over all ordinals.

Given a group $G \in EA$, define the *elementary class* of G, $\mathrm{c}(G)$ as the smallest ordinal α such that $G \in EG_\alpha$. The following three observations are quite elementary.

Lemma 3.7. *For any group G, $\mathrm{c}(G)$ is a non-limit ordinal or 0. If G is countable, then $\mathrm{c}(G) < \omega_1$. If G is finitely generated, then $\mathrm{c}(G)$ is 0 or a successor of a non-limit ordinal.*

Proof. The first claim obviously follows from the definition of EG_α for a limit ordinal α.

Further suppose that G is countable and $\mathrm{c}(G) = \omega_1 + 1$. Note that G cannot split as an extension

$$1 \to N \to G \to Q \to 1,$$

where $\mathrm{c}(N) < \omega_1$ and $\mathrm{c}(Q) < \omega_1$. Indeed otherwise $N, Q \in EG_\beta$ for some $\beta < \omega_1$ and hence $\mathrm{c}(G) \le \beta_1 + 1 < \omega_1$. Thus G is a direct union of a family of groups $\mathcal{H} = \{H_i\}_{i\in I}$ such that $\mathrm{c}(H_i) = \beta_i$ is countable. We enumerate the group $G = \{1, g_1, g_2, \ldots\}$ and choose a chain $\{H_0, H_1, \ldots\} \subseteq \mathcal{H}$ as follows. Let H_0 be any subgroup from $\mathcal{H}$. If H_i is already chosen, let H_{i+1} be any subgroup from $\mathcal{H}$ that contains H_i and g_i. Obviously $G = \bigcup_{i=0}^\infty H_i$ and $\beta = \sup\limits_{i\in\mathbb{N}\cup\{0\}} \beta_i$ is countable. Consequently, $\mathrm{c}(G) \le \beta + 1 < \omega_1$ and we get a contradiction again.

Thus every group G with $\mathrm{c}(G) = \omega_1 + 1$ is uncountable. Using the same arguments as in the previous paragraph, it is now easy to show by transfinite induction that for every ordinal $\alpha \ge 1$, every group G with $\mathrm{c}(G) = \omega_1 + \alpha$ is uncountable.

Finally suppose that G is finitely generated and $\mathrm{c}(G) = \alpha + 1$, where α is limit. As above we can show that G cannot split as an extension of a group from EG_α by a group from EG_α. Hence G is a direct union of groups $\{H_i\}_{i\in I}$ from EG_α. Since G is finitely generated, $G = H_i$ for some i and hence $c(G) \le \alpha$. A contradiction. $\square$

In what follows, we denote by G^ω the direct product of countably many copies of G.

Proposition 3.8. *Let α be a countable limit ordinal.*

(a) There exists a countable group H such that $\mathrm{c}(H) = \alpha + 1$ and for every $K \in \mathcal{E}(H)$ we have $\mathrm{c}(K) \le \alpha + 1$.

(b) For every $n \in \mathbb{N}$, $n \geq 2$, there exists a finitely generated group L such that $\mathrm{c}(L) = \mathrm{c}(L^\omega) = \alpha + n$.

Proof. We first remark that for every $n \in \mathbb{N}$, there exists an elementary amenable group of class n. Indeed it is easy to see that non-cyclic free solvable groups of derived length 2^n have elementary class n for every $n \in \mathbb{N}$. Thus if α is a limit ordinal, by induction we can find ordinals $\beta_1, \beta_2, \ldots$ indexed by natural numbers such that $\sup_i \beta_i = \alpha$ and groups $H_1, H_2, \ldots$ such that $\mathrm{c}(H_i) = \beta_i$ for any $i \in \mathbb{N}$. Let $H = \prod_{i=1}^{\infty} H_i$ be the direct product of H_i's. Then H is a direct union of groups $\prod_{i=1}^{n} H_i$ which all have classes less than α. Hence $\mathrm{c}(H) \leq \alpha + 1$. On the other hand, by the first part of Lemma 3.6, we have $\mathrm{c}(H) \geq c(H_i)$ for every $i \in \mathbb{N}$ and therefore $\mathrm{c}(H) \geq \alpha$. Now Lemma 3.7 implies that $\mathrm{c}(H) = \alpha + 1$.

Let us show that $\mathrm{c}(K) \leq \alpha + 1$ for every $K \in \mathcal{E}(H)$. As K is the direct union of its finitely generated subgroups, it suffices to show that every finitely generated subgroup S of a direct power of H satisfies $\mathrm{c}(S) \leq \alpha$. Since S is finitely generated, it is a subgroup of a direct product of finitely many H_i's. The later product has elementary class less than α and hence $\mathrm{c}(S) < \alpha$ by the first part of Lemma 3.6. This completes the proof of (a).

To prove (b) we have to consider two cases. First assume that $n = 2$. Let H be the countable group of elementary class $\alpha + 1$ provided by the first part of the proposition. Let G be the finitely generated group containing H from Corollary 1.3. Since $G_1 \cap H = \{1\}$, H also embeds in $L = G/G_1$ and the later group is an extension of $G_2/G_1 \in \mathcal{E}(H)$ by a solvable group G/G_2 of derived length 3. In particular, we have $\mathrm{c}(G_2/G_1) \leq \alpha + 1$ and hence $\mathrm{c}(L) \leq \alpha + 2$. On the other hand, $\mathrm{c}(L) \geq \mathrm{c}(H) \geq \alpha + 1$ and $\mathrm{c}(L)$ cannot equal $\alpha + 1$ by Lemma 3.7 as L is finitely generated. Thus $\mathrm{c}(L) = \alpha + 2$. Further we note that L^ω also splits as an extension of a group from $\mathcal{E}(H)$ by a solvable group of derived length 3 and hence $\mathrm{c}(L^\omega) = \alpha + 2$.

Further suppose that $n \geq 3$. By induction, there exists a finitely generated group L_0 such that $\mathrm{c}(L_0) = \mathrm{c}(L_0^\omega) = \alpha + n - 1$. Let $L = L_0 \,\mathrm{wr}\, L_0$. Obviously

$$\alpha + n - 1 = \mathrm{c}(L_0) \leq \mathrm{c}(L) \leq \alpha + n. \tag{19}$$

We want to show that, in fact, $\mathrm{c}(L) = \alpha + n$. Indeed suppose that $\mathrm{c}(L) = \alpha + n - 1$. Then L can be obtained from groups of elementary class at most $\alpha + n - 2$ by applying (E) or (U) once.

Suppose first that L splits as

$$1 \to N \to L \to Q \to 1,$$

where $N, Q \in EG_{\alpha+n-2}$. Then Q cannot contain a subgroup isomorphic to L_0 and hence the intersection of N with the active copy of the group L_0 in $L_0 \,\mathrm{wr}\, L_0$ is nontrivial. Let a be a non-trivial element from this intersection and let B be the image of the canonical embedding of L_0 in the base subgroup of $L_0 \,\mathrm{wr}\, L_0$. Since N is normal in L, it contains the subgroup

$$D = [a, B] = \langle aba^{-1}b^{-1} \mid b \in B \rangle.$$

Obviously D is isomorphic to a subgroup of $L_0 \times L_0$ that surjects onto L_0. Therefore, by the first part of Lemma 3.6 we have

$$\mathrm{c}(N) \geq \mathrm{c}(D) \geq \mathrm{c}(L_0) = \alpha + n - 1,$$

which contradicts our assumption. Similarly if L is a direct union of groups $\{M_i\}_{i \in I}$ from $EG_{\alpha+n-2}$, then $L = M_i$ for some $i \in I$ as L is finitely generated. Hence $\mathrm{c}(L) \leq \alpha + n - 2$, which contradicts (19). Thus we get a contradiction in both cases and hence $\mathrm{c}(L) = \alpha + n$.

It remains to show that $\mathrm{c}(L^\omega) = \alpha + n$. However this is obvious since L^ω splits as an extension of a group isomorphic to L_0^ω by a group isomorphic to L_0^ω, which implies

$$\mathrm{c}(L^\omega) \leq \mathrm{c}(L_0^\omega) + 1 = \alpha + n.$$

□

Proof of Corollary 1.6. The corollary follows from Lemma 3.7, Proposition 3.8, and the observation about solvable groups made at the beginning of the proof of Proposition 3.8. □

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References

[1] G. Arzhantseva, C. Drutu, M. Sapir, Compression functions of uniform embeddings of groups into Hilbert and Banach spaces, *J. Reine Angew. Math.* **633** (2009), 213-235.

[2] G. Arzhantseva, V. Guba, M. Sapir, Metrics on diagram groups and uniform embeddings in a Hilbert space, *Comment. Math. Helv.* **81** (2006), no. 4, 911-929.

[3] T. Austin, Amenable groups with very poor compression into Lebesgue spaces, *Duke Math. J.* **159** (2011), no. 2, 187-222.

[4] M. Brin, Elementary amenable subgroups of R. Thompson's group F, *Int. J. Alg. Comp.* **15** (2005), no. 4, 619-642.

[5] N. Brown, E. Guentner, Uniform embeddings of bounded geometry spaces into reflexive Banach spaces, *Proc. Amer. Math. Soc.* **133** (2005), no. 7, 2045–2050.

[6] C. Chou, Elementary amenable groups, *Ill. J. Math* **24** (1980), no. 3, 396-407.

[7] M. Dadarlat, E. Guentner, Constructions preserving Hilbert space uniform embeddability of discrete groups, *Trans. AMS* **355** (2003), no. 8, 3253-3275.

[8] A. Erschler, On isoperimetric profiles of finitely generated groups, *Geom. Dedicata* **100** (2003), 157-171.

[9] A. Erschler, Piecewise automatic groups, *Duke Math. J.* **134** (2006), no. 3, 591-613.

[10] P. Hall, On the embedding of a group in a join of given groups, Collection of articles dedicated to the memory of Hanna Neumann, VIII. *J. Austral. Math. Soc.* **17** (1974), 434-495.

[11] M. Gromov, Asymptotic Invariants of Infinite Groups, Geometric Group Theory(vol. 2), G. A. Niblo, M. A. Roller (eds), Proc. of the Symposium held in Sussex, LMS Lecture Notes Series **181**, Cambridge University Press 1991.

[12] M. Gromov, Entropy and isoperimetry for linear and non-linear group actions, *Groups Geom. Dyn.* **2** (2008), no. 4, 499-593.

[13] M. Gromov, Random walk in random groups, *Geom. Funct. Anal.* **13** (2003), no. 1, 73-146.

[14] E. Guentner and J. Kaminker, Exactness and uniform embeddability of discrete groups, J. London Math. Soc. (2) **70** (2004), no. 3, 703-718.

[15] E. Guentner, R. Tessera, G. Yu, A notion of geometric complexity and its application to topological rigidity, arXiv:1008.0884

[16] G. Higman, B.H. Neumann, H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* **24** (1949), 247254.

[17] G. Kasparov, G. Yu, The coarse geometric Novikov conjecture and uniform convexity, *Adv. Math.*, **206** (1), (2006), 1-56.

[18] V. Lafforgue, Un renforcement de la propriété (T), *Duke Math. J.* **143** (2008), no. 3, 559-602.

[19] W.Magnus, On a theorem of Marshall Hall, *Ann. Math.* **40** (1939), 764-768.

[20] J. von Neumann, Zur allgemeinen Theorie des Masses, *Fund. Math.* **13** (1929), 73-116.

[21] B.H. Neumann, H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* **34** (1959), 465-479.

[22] A.Yu.Olshanskii, Distortion functions for subgroups, in "Geometric Group Theory Down Under", Proc. of a Special Year in Geometric Group Theory, Canberra, Australia, 1996, Walter de Gruyter, Berlin - New York, 1999, 281-291.

[23] R.E. Phillips, Embedding methods for periodic groups, *Proc. London Math. Soc.* (3) **35** (1977), 238-256.

[24] D. A. Ramras, R. Tessera, G. Yu, Finite decomposition complexity and the integral Novikov conjecture for higher algebraic K-theory, arXiv:1111.7022.

[25] V.N. Remeslennikov, V.G. Sokolov, Certain properties of the Magnus imbedding (Russian), *Algebra i Logika* **9** (1970), 566-578.

[26] R. Tessera, Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces, *Comment. Math. Helv.* **86** (2011), no. 3, 499-535

[27] A. Vershik, Amenability and approximation of infinite groups. Selected translations. *Selecta Math. Soviet.* **2** (1982), no. 4, 311-330.

[28] J. Wilson, Embedding theorems for residually finite groups, *Math. Z.* **174** (1980), no. 2, 149-157.

[29] G. Yu, The Coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert spaces, *Invent. Math.* **139** (2000), 201-240.

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